National University of Science and Technology POLITEHNICA
Bucharest

PhD Dissertation

Functions of generalized Fourier multipliers, α-windowed Fourier transform, localization operators

PhD Supervisor:
Conf. univ. dr. Viorel CATANĂ

Author:
Ioana-Maria FLONDOR

2025
Bucharest

Thesis content

Thesis summary

Please note that the numbering of all the statements in this abstract coincides with the numbering of the PhD thesis.

The dissertation is organized as follows.

In the first chapter of this thesis, entitled "Functions of generalized Fourier multipliers", we introduce and study a class of linear operators defined on a separable Hilbert space $\mathcal{H}$. All these results have been published in the original paper [9], entitled **"Generalized Fourier multipliers"**, a paper that was published in Annals of Functional Analysis, vol. 14, article number 34, 2023, to which the author of the thesis was coauthor with Viorel Catană and Horia-George Georgescu.

Let $\mathcal{H}$ be a separable complex Hilbert space endowed with the inner product $(\cdot,\cdot)_{\mathcal{H}}$, $\{e_n\}_{n\in\mathbb{Z}}$ be an orthonormal basis of $\mathcal{H}$ and $A:\mathcal{H}\to\mathcal{H}$ be a bounded linear operator. Then we introduce the linear operator

$$\mathcal{F}^A_{\mathcal{H}}:\mathcal{H}\to L^2(\mathbb{Z}),\quad \mathcal{F}^A_{\mathcal{H}}f=\{(Af,e_n)_{\mathcal{H}}\}_{n\in\mathbb{Z}} \tag{1}$$

for all $f\in\mathcal{H}$. This operator can be called „coordinatization map".

Theorem 1.1 (Plancherel's Theorem) *Let $A\in\mathcal{B}(\mathcal{H})$ such that $A^2=I$, where $I:\mathcal{H}\to\mathcal{H}$ is the identity operator. Then, the linear operator $\mathcal{F}^A_{\mathcal{H}}:\mathcal{H}\to L^2(\mathbb{Z})$ satisfies the following relation*

$$\left(\mathcal{F}^A_{\mathcal{H}}f,\mathcal{F}^A_{\mathcal{H}}g\right)_{L^2(\mathbb{Z})}=(Af,Ag)_{\mathcal{H}}.$$

for all $f,g\in\mathcal{H}$. Moreover, the linear operator $\mathcal{F}^A_{\mathcal{H}}:\mathcal{H}\to L^2(\mathbb{Z})$ is a bijection.

Remark 1.1 If we suppose in addition that A is a self-adjoint operator, then it follows that $\mathcal{F}^A_{\mathcal{H}}:\mathcal{H}\to L^2(\mathbb{Z})$ is an isometric isomorphism.

Corollary 1.1 The linear operator $\left(\mathcal{F}^A_{\mathcal{H}}\right)^{-1}=\mathcal{F}^A_{\mathbb{Z}}:L^2(\mathbb{Z})\to\mathcal{H}$ defined by

$$\left(\mathcal{F}^A_{\mathcal{H}}\right)^{-1}a=\mathcal{F}^A_{\mathbb{Z}}a=\sum_{n\in\mathbb{Z}}a_nAe_n, \tag{2}$$

for all $a=\{a_n\}_{n\in\mathbb{Z}}$ in $L^2(\mathbb{Z})$, is the inverse of the operator $\mathcal{F}^A_{\mathcal{H}}:\mathcal{H}\to L^2(\mathbb{Z})$.

Remark 1.2 (i) Let $a\in L^2(\mathbb{Z})$ and let $A^2=I$, where $I:\mathcal{H}\to\mathcal{H}$ is the identity operator. Then by the preceding definition $\mathcal{F}^A_{\mathbb{Z}}=\left(\mathcal{F}^A_{\mathcal{H}}\right)^{-1}:L^2(\mathbb{Z})\to\mathcal{H}$ we get $\mathcal{F}^A_{\mathbb{Z}}a=\sum_{n\in\mathbb{Z}}a_nAe_n$. So,

$$A\mathcal{F}^A_{\mathbb{Z}}a=\sum_{n\in\mathbb{Z}}a_nA^2e_n=\sum_{n\in\mathbb{Z}}a_ne_n$$

and

$$\left(A\mathcal{F}^A_{\mathbb{Z}}a,e_m\right)_{\mathcal{H}}=a_m,$$

for all $m\in\mathbb{Z}$. In addition, let us observe that

$$\left\| A\mathcal{F}_{\mathbb{Z}}^{A} a \right\|_{\mathcal{H}}^{2} = \left\| \sum_{n \in \mathbb{Z}} a_n e_n \right\|_{\mathcal{H}}^{2} = \sum_{n \in \mathbb{Z}} |a_n|^2.$$

(ii) Let $A \in \mathcal{B}(\mathcal{H})$ be a bounded linear operator such that $A^2 = I$. Then the following statements are equivalent:
(j) A is a self-adjoint operator;
(jj) A is a unitary operator;
(jjj) A is an isometry.

Remark 1.3 If we consider $A = I : \mathcal{H} \to \mathcal{H}$ and $\mathcal{H} = L^2(S^1)$, where S^1 is the unitary circle with center at the origin, $e_n(s) = \dfrac{1}{\sqrt{2\pi}} e^{ins}$, for $s \in S^1, n \in \mathbb{Z}$ and $\sigma \in L^\infty(\mathbb{Z})$, then we recover the Fourier multipliers on S^1, which have been studied in papers [23], [24], [25], [26] and [35].

In the following we present the definition of generalized Fourier multipliers.

Definition 1.1 Let $\sigma : \mathbb{Z} \to \mathbb{C}$ be a measurable function on $\mathbb{Z}$ and let $A \in \mathcal{B}(\mathcal{H})$ be a bounded linear operator on $\mathcal{H}$. Then, we can define formally the linear operator $T_\sigma^A : \mathcal{H} \to \mathcal{H}$ by the following formula

$$T_\sigma^A f = \sum_{n \in \mathbb{Z}} \sigma(n) (Af, e_n)_{\mathcal{H}} A e_n. \tag{3}$$

for all $f \in \mathcal{H}$.

If we suppose $A^2 = I$ where $I : \mathcal{H} \to \mathcal{H}$ is the identity operator, using Theorem 1.1 and the relations (1), (2), we can rewrite the above operator in the following way

$$T_\sigma^A f = \mathcal{F}_{\mathbb{Z}}^A (\sigma \cdot \mathcal{F}_{\mathcal{H}}^A f) = \left(\mathcal{F}_{\mathcal{H}}^A\right)^{-1} (\sigma \cdot \mathcal{F}_{\mathcal{H}}^A f), \tag{4}$$

for all $f \in \mathcal{H}$.

Let us observe that the first equality in the relation (4) occurs for any bounded linear operator and the second one is valid only if $A^2 = I$.

Let us remark that if $A^2 = I$ and A is a self-adjoint operator, then the generalized Fourier multiplier T_σ^A coincides with a standard Fourier multiplier on $\mathcal{H}$, which has the form

$$T_\sigma = \sum_{n \in \mathbb{Z}} \sigma(n) (f, \varphi_n)_{\mathcal{H}} \varphi_n, \quad \forall f \in \mathcal{H},$$

where $\{\varphi_n\}_{n \in \mathbb{Z}}$ is an orthonormal basis on $\mathcal{H}$.

In particular, if $A = I$ then the generalized Fourier multiplier T_σ^A defined in (3) also coincides with a standard Fourier multiplier. Therefore, we shall call T_σ^A the generalized Fourier multiplier on $\mathcal{H}$ corresponding to the symbol σ, whenever the series in (3) is convergent in $\mathcal{H}$.

We may also remark that, in the particular case $A = I$, where $I : \mathcal{H} \to \mathcal{H}$ is the identity operator and the Hilbert space $\mathcal{H}$ is equal to $L^2(S^1)$ or $L^2(S)$, we obtain two significant examples of standard Fourier multipliers introduced and studied in the papers [7] and [8]. We note that here S^1 represents the unitary circle with center at the origin and $(S, \mathcal{B}, m)$ is a finite measure space so that $L^2(S)$ is a separable Hilbert space.

In the following we are interested in studying the boundedness, compactness and Schatenn von-Neuman properties for this class of generalized Fourier multipliers.

Proposition 1.2 *Let $\sigma : \mathbb{Z} \to \mathbb{C}$ be a measurable function such that $\sigma \in L^1(\mathbb{Z})$ and let $A \in \mathcal{B}(\mathcal{H})$ be a bounded linear operator. Then the generalized Fourier multiplier $T_\sigma^A : \mathcal{H} \to \mathcal{H}$ is bounded. In addition*

$$\left\|T_\sigma^A\right\|_{\mathcal{B}(\mathcal{H})} \leq \|A\|_{\mathcal{B}(\mathcal{H})}^2 \|\sigma\|_{L^1(\mathbb{Z})}.$$

Proposition 1.3 *Let $\sigma \in L^\infty(\mathbb{Z})$ and let $A : \mathcal{H} \to \mathcal{H}$ be a bounded linear operator. Then $T_\sigma^A : \mathcal{H} \to \mathcal{H}$ is a bounded linear operator and*

$$\|T_\sigma^A\|_{\mathcal{B}(\mathcal{H})} \leq \|A\|_{\mathcal{B}(\mathcal{H})}^2 \|\sigma\|_{L^\infty(\mathbb{Z})}.$$

Proposition 1.4 *Let $\sigma : \mathbb{Z} \to \mathbb{C}$ be a measurable function and let $A : \mathcal{H} \to \mathcal{H}$ be a bounded linear operator such that $A^2 = I$. Then $T_\sigma^A : \mathcal{H} \to \mathcal{H}$ is a bounded linear operator if and only if $\sigma \in L^\infty(\mathbb{Z})$. Moreover, if $\sigma \in L^\infty(\mathbb{Z})$, then*

$$\|T_\sigma^A\|_{\mathcal{B}(\mathcal{H})} \leq \|A\|_{\mathcal{B}(\mathcal{H})}^2 \|\sigma\|_{L^\infty(\mathbb{Z})}.$$

In addition, if we suppose that $A : \mathcal{H} \to \mathcal{H}$ is also a self-adjoint operator, then

$$\|T_\sigma^A\|_{\mathcal{B}(\mathcal{H})} = \|A\|_{\mathcal{B}(\mathcal{H})}^2 \|\sigma\|_{L^\infty(\mathbb{Z})} = \|\sigma\|_{L^\infty(\mathbb{Z})}.$$

The last equality is a consequence of Remark 1.2 (iii).

Theorem 1.3 *Let $\sigma \in L^p(\mathbb{Z}), 1 < p < \infty$ and let $A : \mathcal{H} \to \mathcal{H}$ be a bounded linear operator. Then, there exists a unique bounded linear operator $T_\sigma^A : \mathcal{H} \to \mathcal{H}$ such that*

$$\left\|T_\sigma^A\right\|_{\mathcal{B}(\mathcal{H})} \leq \|\sigma\|_{L^p(\mathbb{Z})} \|A\|_{\mathcal{B}(\mathcal{H})}^2,$$

where T_σ^A is given by (4) for $f \in \mathcal{H}$ and all simple function σ on $\mathbb{Z}$ for which $\mu\{n \in \mathbb{Z} : \sigma(n) \neq 0\} < \infty$.

Remark 1.4 In addition, if we suppose $A^2 = I$ it follows that

$$\|T_\sigma^A\|_{\mathcal{B}(\mathcal{H})} \leq \|\sigma\|_{L^p(\mathbb{Z})}$$

by using Remark 1.2 (iii).

In the next theorem we present a result on the spectral theory of generalized Fourier multipliers on $\mathcal{H}$.

Theorem 1.4 *Let $\sigma : \mathbb{Z} \to \mathbb{C}$ be a measurable function such that $\sigma \in L^\infty(\mathbb{Z})$ and let $A : \mathcal{H} \to \mathcal{H}$ be a bounded linear operator such that $A^2 = I$, where $I : \mathcal{H} \to \mathcal{H}$ is the identity operator. Then, $\sigma(n)$ is an eigenvalue of $T_\sigma^A : \mathcal{H} \to \mathcal{H}$ and Ae_n is the corresponding eigenfunction for all $n \in \mathbb{Z}$. Moreover, the spectrum $\Sigma(T_\sigma^A)$ of T_σ^A is given by*

$$\Sigma\left(T_\sigma^A\right) = \{\sigma(n) : n \in \mathbb{Z}\}^c,$$

where $\{\dots\}^c$ *denotes the closure in* $\mathbb{C}$ *of the set* $\{\dots\}$.

Theorem 1.5 *Let* $\sigma : \mathbb{Z} \to \mathbb{C}$ *be a measurable function such that* $\sigma \in L^1(\mathbb{Z})$ *and let* $A : \mathcal{H} \to \mathcal{H}$ *be a bounded and self-adjoint operator. Then the generalized Fourier multiplier* $T_\sigma^A : \mathcal{H} \to \mathcal{H}$ *is in Schatten-von Neumann class* S_1 *(trace class) and*

$$\left\|T_\sigma^A\right\|_{S_1} \leq 4\|A\|^2_{\mathcal{B}(\mathcal{H})}\|\sigma\|_{L^1(\mathbb{Z})}.$$

Theorem 1.6 *Let* $\sigma \in L^p(\mathbb{Z}), 1 \leq p \leq \infty$ *and let* $A : \mathcal{H} \to \mathcal{H}$ *be a bounded and self-adjoint operator. Then the generalized Fourier multiplier* $T_\sigma^A : \mathcal{H} \to \mathcal{H}$ *is in Schatten-von Neumann class* S_p *and*

$$\left\|T_\sigma^A\right\|_{S_p} \leq 4^{\frac{1}{p}}\|A\|^2_{\mathcal{B}(\mathcal{H})}\|\sigma\|_{L^p(\mathbb{Z})}.$$

The following proposition gives a characterization of compact generalized Fourier multipliers $T_\sigma^A : \mathcal{H} \to \mathcal{H}$.

Proposition 1.5 *Let* $\sigma \in L^p(\mathbb{Z}), 1 \leq p < \infty$ *be a measurable function and let* $A : \mathcal{H} \to \mathcal{H}$ *be a bounded and self-adjoint operator. Then the generalized Fourier multiplier* $T_\sigma^A : \mathcal{H} \to \mathcal{H}$ *is compact.*

The following result gives the formula for the trace $tr(T_\sigma^A)$ of the generalized Fourier multiplier $T_\sigma^A : \mathcal{H} \to \mathcal{H}$ when $\sigma \in L^1(\mathbb{Z})$.

Theorem 1.7 *Let* $\sigma \in L^1(\mathbb{Z})$ *and let* $A : \mathcal{H} \to \mathcal{H}$ *be a bounded and self-adjoint operator. Then the generalized Fourier multiplier* $T_\sigma^A : \mathcal{H} \to \mathcal{H}$ *is in Schatten-von Neumann class* S_1 *and its trace is given by*

$$tr(T_\sigma^A) = \sum_{n\in\mathbb{Z}} \sigma(n)\|Ae_n\|^2_{\mathcal{H}},$$

where $\{e_n\}_{n\in\mathbb{Z}}$ *is an orthonormal basis.*

We further define the function of generalized Fourier multipliers $F(T_\sigma^A) : \mathcal{H} \to \mathcal{H}$, where $F : \mathbb{C} \to \mathbb{C}$ is a continuous function.

Definition 1.2 *Let* $F : \mathbb{C} \to \mathbb{C}$ *be a continuous function, let* $\sigma : \mathbb{Z} \to \mathbb{C}$ *be a measurable function and let* $A : \mathcal{H} \to \mathcal{H}$ *be a bounded linear operator. Then we define formally the linear operator* $F(T_\sigma^A) : \mathcal{H} \to \mathcal{H}$ *by the following formula*

$$F(T_\sigma^A)f = \sum_{n\in\mathbb{Z}} F(\sigma(n))(Af, e_n)_{\mathcal{H}}Ae_n, \quad f \in \mathcal{H},$$

where $T_\sigma^A : \mathcal{H} \to \mathcal{H}$ *is a generalized Fourier multiplier on* $\mathcal{H}$.

The following theorem gives necessary and sufficient conditions in order that the function of generalized Fourier multiplier $F(T_\sigma^A) : \mathcal{H} \to \mathcal{H}$ on a Hilbert space $\mathcal{H}$ to be a compact operator.

Theorem 1.8 *Let $\sigma : \mathbb{Z} \to \mathbb{C}$ be a measurable function, let $F : \mathbb{C} \to \mathbb{C}$ be a continuous function such that $F \circ \sigma \in L^\infty(\mathbb{Z})$ and let $A : \mathcal{H} \to \mathcal{H}$ be a bounded and self-adjoint operator such that $A^2 = I$. Then $F(T_\sigma^A) : \mathcal{H} \to \mathcal{H}$ is a compact operator if and only if*

$$\lim_{|n|\to\infty} F(\sigma(n)) = 0.$$

The following result refers to the absolute value of the function of generalized Fourier multipliers $F(T_\sigma^A) : \mathcal{H} \to \mathcal{H}$.

Theorem 1.9 *Let $\sigma : \mathbb{Z} \to \mathbb{C}$ be a measurable function, let $A : \mathcal{H} \to \mathcal{H}$ be a bounded and self-adjoint operator such that $A^2 = I$ and let $F : \mathbb{C} \to \mathbb{C}$ be a continuous function with the following properties:*
(i) $F \circ \sigma \in L^\infty(\mathbb{Z})$;
(ii) $\overline{F \circ \sigma} = F \circ \overline{\sigma}$;
(iii) $F(\sigma(n)) \to 0$ as $|n| \to \infty$;
(iv) $F(z_1 z_2) = F(z_1) F(z_2)$, for all $z_1, z_2 \in \mathbb{C}$.

Then

$$\left|F(T_\sigma^A)\right| = F(T_{|\sigma|}^A),$$

where $\left|F(T_\sigma^A)\right| = \left(F(T_\sigma^A)^* F(T_\sigma^A)\right)^{\frac{1}{2}}$ denotes the absolute value of the operator $F(T_\sigma^A) : \mathcal{H} \to \mathcal{H}$.

In the following result we study the Schatten-von Neumann property of the function of generalized Fourier multipliers $F\left(T_\sigma^A\right) : \mathcal{H} \to \mathcal{H}$.

Theorem 1.10 *Let $\sigma : \mathbb{Z} \to \mathbb{C}$ be a measurable function, let $A : \mathcal{H} \to \mathcal{H}$ be a bounded and self-adjoint operator such that $A^2 = I$ and let $F : \mathbb{C} \to \mathbb{C}$ be a continuous function with the following properties:*
(i) $F \circ |\sigma| \in L^p(\mathbb{Z}), 1 \le p < \infty$;
(ii) $\overline{F \circ \sigma} = F \circ \overline{\sigma}$;
(iii) $|F \circ \sigma| = |F \circ |\sigma||$;
(iv) $F(z_1 z_2) = F(z_1) F(z_2)$, for all $z_1, z_2 \in \mathbb{C}$.

Then $F\left(T_\sigma^A\right) : \mathcal{H} \to \mathcal{H}$ is in the Schatten-von Neumann class S_p if and only if $F \circ |\sigma| \in L^p(\mathbb{Z}), 1 \le p < \infty$. Moreover, if $F \circ |\sigma| \in L^p(\mathbb{Z})$, then

$$\left\|F\left(T_\sigma^A\right)\right\|_{S_p} = \|F \circ |\sigma|\|_{L^p(\mathbb{Z})},$$

where $\|\cdot\|_{S_p}$ denotes the norm in the Schatten-von Neumann class S_p.

Let $A : \mathcal{H} \to \mathcal{H}$ be a bounded linear operator such that $A^2 = I$ and let $F : \mathbb{R} \to \mathbb{R}_+$ be a continuous function. In the following we give some results regarding the solvability of the equation

$$F\left(T_\sigma^A\right) f + \lambda f = g, \quad f, g \in \mathcal{H}, \tag{5}$$

associated to the linear operator $F\left(T_\sigma^A\right) : \mathcal{H} \to \mathcal{H}$ where λ is a positive real number. We can rewrite the equation (5) as follows:

$$\begin{aligned} F\left(T_\sigma^A\right) f + \lambda f = g &\Leftrightarrow \sum_{n\in\mathbb{Z}} F(\sigma(n))(Af, e_n)_{\mathcal{H}} Ae_n + \lambda \sum_{n\in\mathbb{Z}} (Af, e_n)_{\mathcal{H}} Ae_n = g \\ &\Leftrightarrow \sum_{n\in\mathbb{Z}} \left(F(\sigma(n)) + \lambda\right)(Af, e_n)_{\mathcal{H}} Ae_n = g \\ &\Leftrightarrow \sum_{n\in\mathbb{Z}} \left(F(\sigma(n)) + \lambda\right) \mathcal{F}_{\mathcal{H}}^A f(n) Ae_n = g. \end{aligned} \tag{6}$$

Now, let us suppose that $\sigma : \mathbb{Z} \to \mathbb{R}$ is a real-valued non-negative measurable function and that λ is a positive real number. Using the fact that the linear operator $\mathcal{F}_{\mathcal{H}}^A : \mathcal{H} \to L^2(\mathbb{Z})$ is a bijection (by Theorem 1.1), it follows that for any $g \in \mathcal{H}$ there exists $f \in \mathcal{H}$ such that $\mathcal{F}_{\mathcal{H}}^A f(n) = \dfrac{\mathcal{F}_{\mathcal{H}}^A g(n)}{F(\sigma(n)) + \lambda}$ for all $n \in \mathbb{Z}$, because the sequece $\left\{\dfrac{\mathcal{F}_{\mathcal{H}}^A g(n)}{F(\sigma(n)) + \lambda}\right\} \in L^2(\mathbb{Z})$.

Let $g \in \mathcal{H}$ be an arbitrary element and let us remark that $g = \sum\limits_{n\in\mathbb{Z}} (Ag, e_n)_{\mathcal{H}} Ae_n$. Then, it follows that $f \in \mathcal{H}$ defined by

$$f = \sum_{n\in\mathbb{Z}} \frac{(Ag, e_n)_{\mathcal{H}}}{F(\sigma(n)) + \lambda} Ae_n$$

is the unique solution of the equation (5).

Indeed, by relation (6), we have

$$\begin{aligned} \left(F\left(T_\sigma^A\right) + \lambda\right) f &= \sum_{n\in\mathbb{Z}} \left(F(\sigma(n)) + \lambda\right) \cdot \frac{\mathcal{F}_{\mathcal{H}}^A g(n)}{F(\sigma(n)) + \lambda} Ae_n \\ &= \sum_{n\in\mathbb{Z}} (Ag, e_n)_{\mathcal{H}} Ae_n \\ &= A\left(\sum_{n\in\mathbb{Z}} (Ag, e_n)_{\mathcal{H}} e_n\right) \\ &= A(Ag) = g. \end{aligned}$$

In order to study the regularity of the solutions of equation (5) we need to introduce some convenient Hilbert spaces (for more details on the genesis of these Hilbert spaces, see for example the papers [3], [4] and [16]).

Regarding the regularity of the solution of equation (5), we introduce the space $\widetilde{\mathcal{H}}_\sigma^{A,k}$ as follows: first, consider the Hilbert space

$$L_\sigma^{2,k}(\mathbb{Z}) = \left\{ \{\alpha_n\}_{n\in\mathbb{Z}}, \alpha_n \in \mathbb{C} : \sum_{n\in\mathbb{Z}} |\alpha_n|^2 \left(1 + (\sigma(n))^2\right)^k < \infty \right\}, \quad k \in (0, \infty)$$

in which the inner product is given by

$$(\alpha, \beta)_{L^{2,k}_\sigma(\mathbb{Z})} = \sum_{n\in\mathbb{Z}} \alpha_n \overline{\beta_n} \left(1 + (\sigma(n))^2\right)^k,$$

for all $\alpha = (\alpha_n)_{n\in\mathbb{Z}}, \beta = (\beta_n)_{n\in\mathbb{Z}}, \alpha_n, \beta_n \in \mathbb{C}$. Then we set

$$\begin{aligned}\widetilde{\mathcal{H}}^{A,k}_\sigma &= \left\{ f \in \mathcal{H} : \sum_{n\in\mathbb{Z}} |(Af, e_n)_{\mathcal{H}}|^2 \left(1 + (\sigma(n))^2\right)^k < \infty \right\} \\ &= \left\{ f \in \mathcal{H} : \left\{\mathcal{F}^A_{\mathcal{H}} f(n)\right\}_{n\in\mathbb{Z}} \in L^{2,k}_\sigma(\mathbb{Z}) \right\}, \quad k \in (0, \infty).\end{aligned}$$

The space $\widetilde{\mathcal{H}}^{A,k}_\sigma$ is endowed with the inner product

$$\begin{aligned}(f_1, f_2)_{\widetilde{\mathcal{H}}^{A,k}_\sigma} &= \left(\left\{\mathcal{F}^A_{\mathcal{H}} f_1(n)\right\}_{n\in\mathbb{Z}}, \left\{\mathcal{F}^A_{\mathcal{H}} f_2(n)\right\}_{n\in\mathbb{Z}} \right)_{L^{2,k}_\sigma(\mathbb{Z})} \\ &= \sum_{n\in\mathbb{Z}} \mathcal{F}^A_{\mathcal{H}} f_1(n) \overline{\mathcal{F}^A_{\mathcal{H}} f_2(n)} \left(1 + (\sigma(n))^2\right)^k,\end{aligned}$$

if

$$f_1 = \sum_{n\in\mathbb{Z}} \mathcal{F}^A_{\mathcal{H}} f_1(n) A e_n, \quad f_2 = \sum_{n\in\mathbb{Z}} \mathcal{F}^A_{\mathcal{H}} f_2(n) A e_n.$$

Let us remark that $\widetilde{\mathcal{H}}^{A,k}_\sigma$ is a Hilbert space isometrically isomorphic to $L^{2,k}_\sigma(\mathbb{Z})$ by the isometric isomorphism $G^{A,k}_\sigma : \widetilde{\mathcal{H}}^{A,k}_\sigma \to L^{2,k}_\sigma(\mathbb{Z})$ given by

$$G^{A,k}_\sigma f = \left\{\mathcal{F}^A_{\mathcal{H}} f(n)\right\}_{n\in\mathbb{Z}}.$$

Indeed,

$$\left\|G^{A,k}_\sigma f\right\|^2_{L^{2,k}_\sigma(\mathbb{Z})} = \sum_{n\in\mathbb{Z}} \left|\mathcal{F}^A_{\mathcal{H}} f(n)\right|^2 \left(1 + (\sigma(n))^2\right)^k = \|f\|^2_{\widetilde{\mathcal{H}}^{A,k}_\sigma}.$$

Let $F : \mathbb{R} \to \mathbb{R}_+$ be a non-negative real continuous function. Suppose that there exists a positive constant C such that

$$C \left(1 + \tau^2\right)^{\frac{k}{2}} < F(\tau), \tag{7}$$

for all $\tau \in \Sigma\left(T^A_\sigma\right) = \{\sigma(n) : n \in \mathbb{Z}\}^c$. This is an ellipticity condition on function F.

Now, let us consider the space

$$\widetilde{\mathcal{H}}^A_{F,\sigma} = \left\{ f \in \mathcal{H} : \sum_{n\in\mathbb{Z}} |(Af, e_n)_{\mathcal{H}}|^2 \left(1 + F(\sigma(n))\right)^2 < \infty \right\}$$

with the inner product defined by

$$(f, g)_{\widetilde{\mathcal{H}}^A_{F,\sigma}} = \sum_{n\in\mathbb{Z}} (Af, e_n)_{\mathcal{H}} \overline{(Ag, e_n)_{\mathcal{H}}} \left(1 + F(\sigma(n))\right)^2,$$

for all $f, g \in \mathcal{H}$. By the ellipticity condition (7) the following continuous inclusion holds:

$$\widetilde{\mathcal{H}}^A_{F,\sigma} \hookrightarrow \widetilde{\mathcal{H}}^{A,k}_\sigma.$$

Now, we consider the operator $L^A_{F,\sigma} : \widetilde{\mathcal{H}}^A_{F,\sigma} \to \mathcal{H}, L^A_{F,\sigma} = F(T^A_\sigma) + I$, defined by

$$L^A_{F,\sigma} f = \sum_{n\in\mathbb{Z}} \left(F(\sigma(n)) + 1\right) (Af, e_n)_{\mathcal{H}} A e_n$$

for all $f \in \widetilde{\mathcal{H}}^A_{F,\sigma}$, where $I : \mathcal{H} \to \mathcal{H}$ is the identity operator.

Proposition 1.6 *Let $\sigma : \mathbb{Z} \to \mathbb{C}$ be a real-valued non-negative measurable function, let $A : \mathcal{H} \to \mathcal{H}$ be a bounded linear operator such that $A^2 = I$, let $T^A_\sigma : \mathcal{H} \to \mathcal{H}$ be the corresponding generalized Fourier multiplier and let $F : \mathbb{R} \to \mathbb{R}_+$ be a real-valued non-negative continuous function that satisfies relation (7). Then, for any $g \in \mathcal{H}$, the linear equation*

$$L^A_{F,\sigma} f = g$$

has a unique solution $f \in \widetilde{\mathcal{H}}^A_{F,\sigma}$. Moreover,

$$\|f\|_{\widetilde{\mathcal{H}}^A_{F,\sigma}} = \|Ag\|_{\mathcal{H}}.$$

Remark 1.5 If we suppose, in addition, that the linear operator $A : \mathcal{H} \to \mathcal{H}$ in Proposition 1.6 is also a self-adjoint operator, we obtain $\|f\|_{\widetilde{\mathcal{H}}^A_{F,\sigma}} = \|g\|_{\mathcal{H}}$ for $f \in \widetilde{\mathcal{H}}^A_{F,\sigma}$ and $g \in \mathcal{H}$.

Remark 1.6 By the above hypoteses, the linear equation $F(T^A_\sigma)f + \lambda f = g$ is equivalent to the linear equation $L^A_{G,\sigma} f = h$, where $G = \dfrac{F}{\lambda}$ and $h = \dfrac{g}{\lambda}$. So, by Proposition 1.6 the last equation has a unique solution $f \in \widetilde{\mathcal{H}}^A_{G,\sigma}$ and in addition $\|f\|_{\widetilde{\mathcal{H}}^A_{G,\sigma}} = \|Ah\|_{\mathcal{H}}$.

$$\|f\|_{\widetilde{\mathcal{H}}^A_{G,\sigma}} = \|Ah\|_{\mathcal{H}}.$$

Thus, the initial linear equation has a unique solution $f \in \widetilde{\mathcal{H}}^A_{G,\sigma} = \widetilde{\mathcal{H}}^A_{\frac{F}{\lambda},\sigma}$ with

$$\|f\|_{\widetilde{\mathcal{H}}^A_{\frac{F}{\lambda},\sigma}} = \frac{1}{\lambda}\|Ag\|_{\mathcal{H}}.$$

If the linear operator $A \in \mathcal{B}(\mathcal{H})$ is also a self-adjoint operator, then

$$\|f\|_{\widetilde{\mathcal{H}}^A_{\frac{F}{\lambda},\sigma}} = \frac{1}{\lambda}\|g\|_{\mathcal{H}}.$$

In the following we present an example of a function of generalized Fourier multipliers in connection with a well-known and very important transformation showing the interplay between the time–frequency analysis and pseudo–differential operators, namely the Weyl transform. For more details on the Weyl transform, see for example the papers [36] and [37].

Let $W_\tau : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ be the Weyl transform defined by

$$(W_\tau f, g)_{L^2(\mathbb{R}^n)} = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \tau(x,\xi) W(f,g)(x,\xi) \mathrm{d}x \mathrm{d}\xi$$

where

$$W(f,g)(x,\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-iy\xi} f\left(x + \frac{y}{2}\right) \overline{g\left(x - \frac{y}{2}\right)} \mathrm{d}y$$

is the Wigner transform and

$$\tau \in L^r_*\left(\mathbb{R}^{2n}\right) = \left\{\tau \in L^r\left(\mathbb{R}^{2n}\right) : \hat{\tau} \in L^{r'}\left(\mathbb{R}^{2n}\right)\right\},$$

where $2 \le r \le \infty$ and r' is a conjugate index of r $\left(\text{i.e. } \dfrac{1}{r} + \dfrac{1}{r'} = 1\right)$ and $\hat{\tau}$ is a Fourier transform of τ. Then by Theorem 14.3 from the paper [37], it follows that the Weyl transform $W_\tau : L^2\left(\mathbb{R}^n\right) \to L^2\left(\mathbb{R}^n\right)$ is a bounded linear and compact operator.

When τ is a real valued symbol it follows that $W_\tau : L^2\left(\mathbb{R}^n\right) \to L^2\left(\mathbb{R}^n\right)$ is also a self-adjoint linear operator. Hence, using the spectral theorem for compact and self-adjoint operators, we get

$$W_\tau f = \sum_{n\in\mathbb{Z}} \sigma(n) \left(f, \varphi_n\right) \varphi_n, \qquad f \in L^2\left(\mathbb{R}^n\right),$$

where $\{\varphi_n\}_{n\in\mathbb{Z}}$ is an orthonormal basis for $L^2\left(\mathbb{R}^n\right)$ consisting of eigenvectors of $W_\tau : L^2\left(\mathbb{R}^n\right) \to L^2\left(\mathbb{R}^n\right)$ and $\sigma(n)$ is the eigenvalue of $W_\tau : L^2\left(\mathbb{R}^n\right) \to L^2\left(\mathbb{R}^n\right)$ corresponding to the eigenfunction $\varphi_n, n \in \mathbb{Z}$; the convergence of the series is understood to be in $L^2\left(\mathbb{R}^n\right)$.

Let $A : \mathcal{H} \to \mathcal{H}$ be a self-adjoint invertible operator. Then, for $\{\psi_n\}_{n\in\mathbb{Z}} = \{A^{-1}\varphi_n\}_{n\in\mathbb{Z}}$, we can write

$$W_\tau f = \sum_{n\in\mathbb{Z}} \sigma(n)(f, A\psi_n)A\psi_n = \sum_{n\in\mathbb{Z}} \sigma(n)(Af, \psi_n)A\psi_n = W_\tau^A f,$$

the last equality being a notation.

Let $F : \mathbb{C} \to \mathbb{C}$ be a continuous function. We define the linear operator $F\left(W_\tau^A\right) : L^2\left(\mathbb{R}^n\right) \to L^2\left(\mathbb{R}^n\right)$, by

$$F\left(W_\tau^A\right) f = \sum_{n\in\mathbb{Z}} F\left(\sigma(n)\right) (Af, \psi_n)A\psi_n.$$

Then we can prove similar results for the operator $F(W_\tau^A) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ as those obtained for the operator $F\left(T_\sigma^A\right) : \mathcal{H} \to \mathcal{H}$, where $\mathcal{H} = L^2(\mathbb{R}^n)$ and $T_\sigma^A = W_\tau^A$. For example, we can state the following theorem.

Theorem 1.11 *Let $\tau : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a symbol with the properties stated above, let $W_\tau^A : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ be the corresponding generalized Fourier multiplier and let $F(W_\tau^A) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ be a linear operator, where $A : \mathcal{H} \to \mathcal{H}$ is a bounded and self-adjoint operator such that $A^2 = I$. Let $\sigma : \mathbb{Z} \to \mathbb{C}$ be a measurable function and let $F : \mathbb{C} \to \mathbb{C}$ be a continuous function with the following properties:*
(i) $F(z_1 z_2) = F(z_1)F(z_2)$ for all $z_1, z_2 \in \mathbb{C}$;
(ii) $|F \circ \sigma| = |F \circ |\sigma||$;
(iii) $\overline{F \circ \sigma} = F \circ \sigma$.

Then the following statements are true:
(j) The linear operator $F(W_\tau^A) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is bounded if and only if $F \circ \sigma \in L^\infty(\mathbb{Z})$. Moreover, if $F \circ \sigma \in L^\infty(\mathbb{Z})$ then

$$\|F(W_\tau^A)\|_{\mathcal{B}(L^2(\mathbb{R}^n))} = \|F \circ \sigma\|_{L^\infty(\mathbb{Z})};$$

(jj) Suppose that the linear operator $F(W_\tau^A) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a bounded linear operator. Then $F(W_\tau^A) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is compact if and only if $F(0) = 0$;
(jjj) The linear operator $F(W_\tau^A) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is in the Schatten-von Neumann class $S_p, 1 \le p < \infty$, if and only if $F \circ |\sigma| \in L^p(\mathbb{Z})$. Moreover, if $F \circ |\sigma| \in L^p(\mathbb{Z})$, then

$$\|F(W_\tau^A)\|_{S_p} = \|F \circ |\sigma|\|_{L^p(\mathbb{Z})}.$$

Remark 1.7 If the bounded linear operator $A \in \mathcal{B}(\mathcal{H})$ satisfies the hypotheses in Theorem 1.11, then the Weyl transform is actually a standard Fourier multiplier.

Chapter 2, entitled "α-windowed Fourier transform (α-WFT)", is dedicated to the introduction of a new time-frequency transform, called the α-WFT, $G_\phi^\alpha : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ where α is a fractional parameter and $\phi \in L^2(\mathbb{R}) \setminus \{0\}$ is a window function.

The results presented in this chapter were the subject of the original paper [10], entitled **"α-Windowed Fourier transform (α-WFT)"**, published in Journal of Pseudo-Differential Operators and Applications, vol. 15, article number 75, 2024, to which the author of the thesis was coauthor with Viorel Catană and Mihaela Grațiela Scumpu.

To begin with, we briefly introduce some notions related to the fractional Fourier transform, which was originally introduced by Almeida in his paper [1].

Definition 2.1 *For any function $f \in L^2(\mathbb{R})$, the α-order fractional Fourier transform (FrFT) is denoted by $\mathcal{F}^\alpha$ and is given by*

$$\mathcal{F}^\alpha[f](\xi) = \int_{\mathbb{R}} f(x)\mathcal{K}_\alpha(x,\xi)dx, \tag{8}$$

where $\mathcal{K}_\alpha(x,\xi)$ denotes the kernel of the FrFT and is given by

$$\mathcal{K}_\alpha(x,\xi)=\begin{cases}\sqrt{\frac{1-i\cot\alpha}{2\pi}}\exp\left\{\frac{i(x^2+\xi^2)\cot\alpha}{2}-i\xi x\csc\alpha\right\}, & \alpha\neq k\pi\\ \delta(x-\xi), & \alpha=2k\pi\\ \delta(x+\xi), & \alpha=2(k+1)\pi,\end{cases}$$

where $(x,\xi)\in\mathbb{R}^2$*,* $k\in\mathbb{Z}$ *and* $\delta(\cdot)$ *is a Dirac function.*

In the particular case $\alpha=\dfrac{\pi}{2}$ the fractional Fourier transform of the function $f\in L^2(\mathbb{R})$ reduces, according to [1], to the classical Fourier transform defined by

$$\mathcal{F}[f](\xi)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}f(x)\exp\{-i\xi x\}\,dx.$$

The inverse fractional Fourier transform corresponding to the relation (8) is given by

$$f(x)=\int_{\mathbb{R}}\mathcal{F}^\alpha[f](\xi)\overline{\mathcal{K}_\alpha(x,\xi)}d\xi.$$

For any $f,g\in L^2(\mathbb{R})$, the Parseval's relation for the fractional Fourier transform is given by

$$\left(\mathcal{F}^\alpha[f],\mathcal{F}^\alpha[g]\right)_{L^2(\mathbb{R})}=(f,g)_{L^2(\mathbb{R})}.$$

In particular, for $f=g$, we obtain

$$\|\mathcal{F}^\alpha[f]\|^2_{L^2(\mathbb{R})}=\|f\|^2_{L^2(\mathbb{R})}.$$

which is the energy preserving relation.

The next result is a definition of fractional convolution.

Definition 2.2 *Let* $f,g\in L^2(\mathbb{R})$ *be two functions. Then the* α*-order fractional convolution is denoted by* $*_\alpha$ *and is defined as*

$$(f*_\alpha g)(u)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}f(x)g(u-x)\exp\left\{\frac{i(x^2-u^2)\cot\alpha}{2}\right\}dx. \tag{9}$$

For $\alpha=\dfrac{\pi}{2}$, the fractional convolution defined above reduces to the classical convolution given by

$$(f*g)(u)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}f(x)g(u-x)dx.$$

The convolution theorem corresponding to (9) states that

$$\mathcal{F}^\alpha[f*_\alpha g](\xi)=\mathcal{F}^\alpha[f](\xi)\mathcal{F}[g](\xi\csc\alpha),$$

for all $f,g\in L^2(\mathbb{R})$.

In the following we define the concept of the α-windowed Fourier transform. First, we recall the definition of the classical windowed Fourier transform (WFT) introduced by Gabor in the paper [15].

Definition 2.3 (WFT) *For a window function* $\phi\in L^2(\mathbb{R})\setminus\{0\}$*, its window daughter function or its windowed Fourier kernel is denoted by* $\phi_{\omega,u}$ *and is defined by*

$$\phi_{\omega,u}(x) = \phi(x-u)\exp\{i\omega x\}.$$

The WFT of $f \in L^2(\mathbb{R})$ with respect to the window function $\phi \in L^2(\mathbb{R}) \setminus \{0\}$ is defined by

$$G_\phi f(\omega,u) = \int_{\mathbb{R}} f(x)\overline{\phi_{\omega,u}(x)}dx.$$

For an efficient mathematical treatment of the α-windowed Fourier transform to be introduced, we define a family of functions $\mathcal{F}_\phi^\alpha(\omega,u)$.

Definition 2.4 *For a window function $\phi \in L^2(\mathbb{R}) \setminus \{0\}$ together with the fractional parameter α, a family of functions $\mathcal{F}_\phi^\alpha(\omega,u)$ is defined by*

$$\mathcal{F}_\phi^\alpha(\omega,u) = \left\{\phi_{\omega,u}^\alpha(x) := \phi(x-u)\exp\left\{i\omega x - \frac{i(x^2-u^2)\cot\alpha}{2}\right\}, \omega, u, x \in \mathbb{R}\right\},$$

where $\alpha \neq n\pi, n \in \mathbb{Z}$ is called a fractional parameter.

Lemma 2.1 For $\phi_{\omega,u}^\alpha \in L^2(\mathbb{R})$ we have

$$\left\|\phi_{\omega,u}^\alpha\right\|_{L^2(\mathbb{R})} = \|\phi\|_{L^2(\mathbb{R})}.$$

Definition 2.5 (α-WFT) *Let $\phi \in L^2(\mathbb{R}) \setminus \{0\}$ be a window function and let α be a fractional parameter such that $\alpha \neq n\pi, n \in \mathbb{Z}$. Then, the α-WFT of $f \in L^2(\mathbb{R})$ with respect to ϕ and α is defined by*

$$\begin{aligned} G_\phi^\alpha f(\omega,u) &= \left(f, \phi_{\omega,u}^\alpha\right)_{L^2(\mathbb{R})} \\ &= \int_{\mathbb{R}} f(x)\overline{\phi_{\omega,u}^\alpha(x)}dx \\ &= \int_{\mathbb{R}} f(x)\overline{\phi(x-u)\exp\left\{i\omega x - \frac{i(x^2-u^2)\cot\alpha}{2}\right\}}dx \\ &= \int_{\mathbb{R}} f(x)\overline{\phi(x-u)}\exp\left\{-i\omega x + \frac{i(x^2-u^2)\cot\alpha}{2}\right\}dx, \end{aligned} \tag{10}$$

for all $(\omega,u) \in \mathbb{R}^2$.

Remark 2.1 It is worth noticing that the α-windowed Fourier transform (α-WFT) boils down to the usual windowed Fourier transform (WFT) when $\alpha = \frac{\pi}{2}$.

The properties of the α-windowed Fourier transform were a topic of interest in the second chapter, so we state some results below.

Remark 2.2 The α-windowed Fourier transform defined in the relation (10) can also be expressed in the convolution form as

$$G_\phi^\alpha f(\omega,u) = \left(M_{-\omega}f *_\alpha \tilde{\phi}\right)(u),$$

where $M_{-\omega}f(x) = \exp\{-i\omega x\} f(x)$ și $\tilde{\phi}(x) = \overline{\phi}(-x)$.

Moreover, we can observe that the following equality holds

$$\mathcal{F}^\alpha[G_\phi^\alpha f(\omega,u)](\xi) = \mathcal{F}^\alpha[M_{-\omega} f](\xi)\mathcal{F}[\tilde{\phi}](\xi \csc\alpha).$$

Lemma 2.2 Let $G_\phi^\alpha f(\omega,u)$ be the α-windowed Fourier transform of any function $f \in L^2(\mathbb{R})$ with respect to the window function $\phi \in L^2(\mathbb{R})$. Then, we get

$$G_\phi^\alpha f(\omega,u) = \exp\{-i\omega u\}\int_{\mathbb{R}} \mathcal{F}^\alpha[f](\xi)\overline{\mathcal{F}[\exp\{i\omega z\}\phi(z)]}(\xi\csc\alpha)\overline{\mathcal{K}_\alpha(u,\xi)}d\xi,$$

where $\mathcal{F}$ denotes the Fourier transform given by (8) in the particular case $\alpha = \dfrac{\pi}{2}$ and $(\omega,u) \in \mathbb{R}^2$.

The following two results refer to the boundedness of the α-windowed Fourier transform.

Theorem 2.3 *Let $\phi \in L^p(\mathbb{R}), 1 \le p \le \infty$ and let $f \in L^1(\mathbb{R})$. Then,*

$$\left\| G_\phi^\alpha f(\omega,\cdot)\right\|_{L^p(\mathbb{R})} \le \|\phi\|_{L^p(\mathbb{R})}\|f\|_{L^1(\mathbb{R})},$$

for all $\omega \in \mathbb{R}$.

Proposition 2.1 *Let $\phi \in L^p(\mathbb{R})$ and let $f \in L^q(\mathbb{R})$ where $p,q \in [1;\infty]$ such that $\dfrac{1}{p}+\dfrac{1}{q} = 1$. Then,*

$$\left|G_\phi^\alpha f(\omega,u)\right| \le \|\phi\|_{L^p(\mathbb{R})}\|f\|_{L^q(\mathbb{R})},$$

for all $(\omega,u) \in \mathbb{R}^2$.

Remark 2.3 If we assume $p = q = 2$ under the assumptions of Proposition 2.1 we obtain the estimate

$$\left|G_\phi^\alpha f(\omega,u)\right| \le \|\phi\|_{L^2(\mathbb{R})}\|f\|_{L^2(\mathbb{R})},$$

for all $(\omega,u) \in \mathbb{R}^2$. Thus, it follows that $G_\phi^\alpha h$ is a bounded function on $\mathbb{R}^2$ for all $f,\phi \in L^2(\mathbb{R})$, where $h(x) = f(x)\exp\left\{\frac{ix^2\cot\alpha}{2}\right\}$.

Proposition 2.2 *The α-windowed Fourier transform (α-WFT) of a function $f \in L^2(\mathbb{R})$ with respect to a fractional parameter α and a window function $\phi \in L^2(\mathbb{R})\setminus\{0\}$ can be reduced to the classical windowed Fourier transform (WFT), as follows*

$$G_\phi^\alpha f(\omega,u) = \exp\left\{-\frac{iu^2\cot\alpha}{2}\right\}G_\phi h(\omega,u),$$

for all $(\omega,u) \in \mathbb{R}^2$, where $h(x) = f(x)\exp\left\{\dfrac{ix^2\cot\alpha}{2}\right\}$.

In the next result we study some properties of the α-windowed Fourier transform.

Theorem 2.4 *Let $\phi \in L^2(\mathbb{R})\setminus\{0\}$ be a window function and let α be a fractional parameter. Then, the α-windowed Fourier transform (α-WFT) satisfies the following properties:*
(i) ***Linearity:***

$$[G_\phi^\alpha(\lambda f + \mu g)(\omega, u)] = \lambda G_\phi^\alpha f(\omega, u) + \mu G_\phi^\alpha g(\omega, u),$$

for all $f, g \in L^2(\mathbb{R})$ and for arbitrary constants λ and μ.
(ii) ***Parity***:

$$G_{P\phi}^\alpha(Pf)(\omega, u) = G_\phi^\alpha f(-\omega, -u),$$

where $Pf(x) = f(-x)$ for every $f \in L^2(\mathbb{R})$.
(iii) ***Modulation***:

$$G_\phi^\alpha(M_{\omega_0} f)(\omega, u) = G_\phi^\alpha f(\omega - \omega_0, u),$$

where $M_{\omega_0} f(x) = \exp\{i\omega_0 x\} f(x)$ for every $f \in L^2(\mathbb{R})$.
(iv) ***Shift***:

$$\begin{aligned} G_\phi^\alpha(T_{x_0} f)(\omega, u) &= \exp\{-iux_0 \cot\alpha\} \exp\{-i(\omega - x_0 \cot\alpha)x_0\} \\ &\times G_\phi^\alpha f(\omega - x_0 \cot\alpha, u - x_0), \end{aligned}$$

where $T_{x_0} f(x) = f(x - x_0)$ for every $f \in L^2(\mathbb{R})$.
(v) ***Conjugation***:

$$G_{\overline{\phi}}^\alpha(\overline{f})(\omega, u) = \overline{G_\phi^{-\alpha} f(-\omega, u)},$$

for all $f \in L^2(\mathbb{R})$.
(vi) ***Switching f with ϕ***:

$$G_\phi^\alpha f(\omega, u) = \exp\left\{\frac{-iu^2 \cot\alpha}{2}\right\} \exp\{-i\omega u\} \overline{G_f^{-\alpha} \phi(-\omega + u\cot\alpha, -u)}$$

for all $f \in L^2(\mathbb{R})$.

In the following, we recall a property of the α-windowed Fourier transform concerning the orthogonality relation.

Theorem 2.5 (Orthogonality relation) *Let $\phi, \psi \in L^2(\mathbb{R}) \setminus \{0\}$ be two window functions and let α be a fractional parameter. Then*

$$\int_{\mathbb{R}} \int_{\mathbb{R}} G_\phi^\alpha f(\omega, u) \overline{G_\psi^\alpha g(\omega, u)} d\omega du = 2\pi (\psi, \phi)_{L^2(\mathbb{R})} (f, g)_{L^2(\mathbb{R})},$$

for all $f, g \in L^2(\mathbb{R})$.

Remark 2.6 According to this theorem, the following statements are true:
(i) If $\phi = \psi$, than

$$\int_{\mathbb{R}} \int_{\mathbb{R}} G_\phi^\alpha f(\omega, u) \overline{G_\phi^\alpha g(\omega, u)} d\omega du = 2\pi \|\phi\|_{L^2(\mathbb{R})}^2 (f, g)_{L^2(\mathbb{R})}.$$

(ii) If $f = g$ and $\phi = \psi$, then

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \left|G_\phi^\alpha f(\omega, u)\right|^2 d\omega du = 2\pi \|\phi\|_{L^2(\mathbb{R})}^2 \|f\|_{L^2(\mathbb{R})}^2,$$

or equivalently

$$\left\|G_\phi^\alpha f\right\|_{L^2(\mathbb{R})} = (2\pi)^{\frac{1}{2}} \|\phi\|_{L^2(\mathbb{R})} \|f\|_{L^2(\mathbb{R})}.$$

(iii) If $\|\phi\|_{L^2(\mathbb{R})} = 1$ in (ii), then

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \left|G_\phi^\alpha f(\omega, u)\right|^2 d\omega du = 2\pi \|f\|^2_{L^2(\mathbb{R})}.$$

(iv) If $\|\phi\|_{L^2(\mathbb{R})} = 1$ and $\|f\|_{L^2(\mathbb{R})} = 1$ in (ii), then

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \left|G_\phi^\alpha f(\omega, u)\right|^2 d\omega du = 2\pi.$$

Another interesting property of the windowed Fourier transform is the inversion formula, which allows the reconstruction of the original signal from its windowed Fourier transform.

Theorem 2.6 (**Inversion formula**) *Let $\phi, \psi \in L^2(\mathbb{R})\backslash\{0\}$ be two window functions such that $(\psi, \phi)_{L^2(\mathbb{R})} \neq 0$ and let α be a fractional parameter. Then, any function $f \in L^2(\mathbb{R})$ can be reconstructed as follows*

$$f(x) = \frac{1}{2\pi\,(\psi, \phi)_{L^2(\mathbb{R})}} \int_{\mathbb{R}} \int_{\mathbb{R}} G_\phi^\alpha f(\omega, u) \psi_{\omega,u}^\alpha(x) d\omega du, \tag{11}$$

for all $x \in \mathbb{R}$. Moreover, if $\phi = \psi$, we obtain

$$f(x) = \frac{1}{2\pi \|\phi\|^2_{L^2(\mathbb{R})}} \int_{\mathbb{R}} \int_{\mathbb{R}} G_\phi^\alpha f(\omega, u) \phi_{\omega,u}^\alpha(x) d\omega du$$

for all $x \in \mathbb{R}$.

In the following we present a result on the range characterization of the α-windowed Fourier transform.

Theorem 2.7 (**Characterization of range of G_ϕ^α**) *Let $\phi \in L^2(\mathbb{R}) \setminus \{0\}$ be a window function such that $\|\phi\|_{L^2(\mathbb{R})} = 1$ and let α be a fractional parameter. Suppose that $h \in L^2(\mathbb{R})$. Then $h \in G_\phi^\alpha\left(L^2(\mathbb{R})\right)$ if and only if it satisfies*

$$h(\omega', u') = \int_{\mathbb{R}} \int_{\mathbb{R}} h(\omega, u) \left(\phi_{\omega,u}^\alpha, \phi_{\omega',u'}^\alpha\right)_{L^2(\mathbb{R})} d\omega du,$$

for all $(\omega', u') \in \mathbb{R}^2$.

The admissibility condition associated with the α-windowed Fourier transform (α-WFT) is stated in the following result.

Proposition 2.3 *Let $\phi \in L^2(\mathbb{R}) \setminus \{0\}$ be a window function. Then ϕ is admissible if*

$$0 < C_\phi = \int_{\mathbb{R}} \left|\mathcal{F}\left[\exp\{i\omega\cdot\}\phi(\cdot)\right](\xi \csc\alpha)\right|^2 d\omega < \infty, \quad \text{a.e. } \xi \in \mathbb{R}.$$

In the next proposition we present a convergence result regarding the reconstruction formula (11).

Proposition 2.4 *Let $\phi \in L^2(\mathbb{R}) \setminus \{0\}$ be a window function, let α be a fractional parameter and let $f \in L^2(\mathbb{R})$ such that*

$$f_{M,N}(x) = \frac{1}{2\pi \|\phi\|^2_{L^2(\mathbb{R})}} \int_{\mathbb{R}} \int_{M<\omega<N} G^\alpha_\phi f(\omega, u)\phi^\alpha_{\omega,u}(x) d\omega du.$$

Then, $f_{M,N}$ is uniformly continuous and

$$\mathcal{F}^\alpha[f_{M,N}](\xi) = \frac{1}{2\pi \|\phi\|^2_{L^2(\mathbb{R})}} \mathcal{F}^\alpha[f](\xi) \left\{ \int_{M<\omega<N} \left|\mathcal{F}\left[\exp\{i\omega z\}\phi(z)\right](\xi \csc\alpha)\right|^2 d\omega \right\}.$$

Proposition 2.5 *Let $\phi \in L^2(\mathbb{R})$ be a window function. Then, for any $f \in L^2(\mathbb{R})$, we have*

$$\lim_{M\to-\infty, N\to\infty} \left\| f - \frac{2\pi \|\phi\|^2_{L^2(\mathbb{R})}}{C_\phi} f_{M,N} \right\|_{L^2(\mathbb{R})} = 0.$$

Moreover, in case $\mathcal{F}^\alpha[f] \in L^1(\mathbb{R})$, then

$$\lim_{M\to-\infty, N\to\infty} \left\| f - \frac{2\pi \|\phi\|^2_{L^2(\mathbb{R})}}{C_\phi} f_{M,N} \right\|_{L^\infty(\mathbb{R})} = 0.$$

In signal processing, an uncertainty principle states that the product of signal variances in the time and frequency domains has a lower bound. In recent years, several authors have proposed generalizations of the uncertainty principles for different types of functions and time-frequency transforms (see [18], [19], [34] and [39] for uncertainty principles associated to the canonical linear transform (LCT)).

In the first theorem of subchapter 2.4 we formulate an uncertainty principle associated to the α-windowed Fourier transform (α-WFT defined in the relation (10)). Then, in Theorem 2.9, we briefly present a form of the Lieb uncertainty principle for the α-windowed Fourier transform. For more details regarding the Lieb uncertainty principle, see for example papers [2], [17] and [21].

Theorem 2.8 *Let $G^\alpha_\phi f(\omega, u)$ be the α-windowed Fourier transform of a non trivial function $f \in L^2(\mathbb{R})$. Then the following uncertainty inequality holds*

$$\left\{ \int_{\mathbb{R}} |\xi|^2 \left|\mathcal{F}^\alpha[f](\xi)\right|^2 d\xi \right\}^{\frac{1}{2}} \left\{ \int_{\mathbb{R}} \int_{\mathbb{R}} |\rho|^2 \left|\mathcal{F}^\beta[G^\alpha_\phi f(\omega, u)](\rho)\right|^2 d\omega d\rho \right\}^{\frac{1}{2}} \geq \frac{\pi \|\phi\|^2_{L^2(\mathbb{R})} |\sin(\alpha - \beta)|}{\sqrt{C_\phi}} \|f\|^2_{L^2(\mathbb{R})},$$

where α and β are chosen such that $\beta = \alpha - \gamma$, $\sin\alpha, \sin\beta, \sin\gamma \neq 0$ and ϕ is an admissible wavelet.

Theorem 2.9 (Lieb's inequality for α-WFT) *Let $f \in L^2(\mathbb{R})$ be a function and $\phi \in L^2(\mathbb{R}) \setminus \{0\}$ be a window function. Then,*

$$\left\| G_\phi^\alpha f \right\|_{L^p(\mathbb{R}^2)} \leq \left(\frac{2}{p}\right)^{\frac{1}{p}} \|f\|_{L^2(\mathbb{R})} \|\phi\|_{L^2(\mathbb{R})}$$

where $2 \leq p < \infty$.

In subchapter 2.5 we establish a connection between the α-windowed Fourier transform (α-WFT) and a version of the fractional Wigner distribution defined by analogy with the α-WFT.

Definition 2.6 (β-WD) *Let $\beta \in \mathbb{R}$ be a fractional parameter such that $\beta \neq n\pi, n \in \mathbb{Z}$. Then, the fractional β-Wigner distribution (β-WD) of $f \in L^2(\mathbb{R})$ is defined as*

$$W^\beta f(t,\xi) = \int_{\mathbb{R}} f\left(t + \frac{x}{2}\right) \overline{f\left(t - \frac{x}{2}\right) \exp\left\{i\left(\xi x - \frac{(x^2 - \xi^2)\cot\beta}{2}\right)\right\}} dx, \tag{12}$$

where $(t,\xi) \in \mathbb{R}^2$.

In the following theorem we establish a connection between the β-Wigner fractional distribution (β-WD) and the α-windowed Fourier transform (α-WFT).

Theorem 2.10 *If $G^\alpha f$ and $W^\beta f$ are the α-WFT and β-WD of $f \in L^2(\mathbb{R})$ defined by (10) and (12), respectively, then the following equality holds*

$$\begin{aligned} W^\beta f(t,\xi) = & \frac{1}{\pi \|\phi\|^2_{L^2(\mathbb{R})}} \exp\left\{i\left(-4t^2\cot\alpha - 2\xi t + \frac{\xi^2}{2}\cot\beta\right)\right\} \\ & \times \int_{\mathbb{R}}\int_{\mathbb{R}} \exp\left\{2it(u\cot\alpha + \omega)\right\} \overline{G^\alpha_{\tilde{\tilde{\phi}}}(M_{-2t\cot\alpha} f)}(-\omega, -(u-2t)) \\ & \times G_\phi^\alpha\left(M_{-2\xi\cdot+2(\cdot-t)^2\cot\beta} f\right)(\omega,u) d\omega du. \end{aligned}$$

In the last subchapter of Chapter 2 we introduce the semi-discrete and discrete versions of the α-windowed Fourier transform (α-WFT). For the semi-discrete version we fix the translation parameter u and let the frequency parameter ω vary on the discrete scales. We then consider both parameters varying on a discrete grid in the fractional time-frequency plane. Finally, we present the reconstruction formula associated to the discret α-windowed Fourier transform.

(i) *The semi-discrete version of the α-WFT*

For this case, we consider $\omega = m\omega_0$ where ω_0 is a fixed positive constant (named a lattice parameter) and $m \in \mathbb{Z}$. Then, the continuous family $\mathcal{F}_\phi^\alpha(\omega,u)$ from Definition 2.4 becomes

$$\mathcal{F}_\phi^\alpha(m,u) = \left\{\phi^\alpha_{m,u}(x) := \phi(x-u)\exp\left\{im\omega_0 x - \frac{i(x^2-u^2)\cot\alpha}{2}\right\}, x \in \mathbb{R}\right\}, \tag{13}$$

where $\alpha \neq n\pi, n \in \mathbb{Z}$ și $(m,u) \in \mathbb{Z} \times \mathbb{R}$.

Now we can give a definition of the semi-discrete α-WFT.

Definition 2.7 *Let $\phi \in L^2(\mathbb{R}) \setminus \{0\}$ be a window function and let α be a fractional parameter. Then, for any $f \in L^2(\mathbb{R})$, we have*

$$G_\phi^\alpha f(m,u) = \int_{\mathbb{R}} f(x)\overline{\phi_{m,u}^\alpha(x)}dx,$$

where $\phi_{m,u}^\alpha$ is given by (13) and the integer m controls the frequency ω.

(ii) *The discrete version of the α-WFT*

For this version, we consider $\omega = m\omega_0$ and $u = nu_0$ where ω_0 and u_0 are fixed positive constants (called lattice parameters) and $m, n \in \mathbb{Z}$. In this context, we obtain the following discrete family of functions

$$\phi_{m,n}^\alpha(x) = \phi(x - nu_0) \exp\left\{ im\omega_0 x - \frac{i\left(x^2 - (nu_0)^2\right)\cot\alpha}{2} \right\}, \tag{14}$$

where $\alpha \neq k\pi, k \in \mathbb{Z}$ și $x \in \mathbb{R}$.

Now we can give a definition of the discrete α-WFT.

Definition 2.8 *Let $\phi \in L^2(\mathbb{R}) \setminus \{0\}$ be a window function and let α be a fractional parameter. Then, for any $f \in L^2(\mathbb{R})$, we have*

$$G_\phi^\alpha f(m,n) = \int_{\mathbb{R}} f(x)\overline{\phi_{m,n}^\alpha(x)}dx,$$

where $\phi_{m,n}^\alpha$ is given by (14) and the integers m and n control the frequency and the translation, respectively.

Chapter 3 is entitled ”Localization operators associated to the α-windowed Fourier transform”. The first section of this chapter is dedicated to the localization operators related to the α-windowed Fourier transform, the results related to this topic being included in the original paper [11], entitled **“Localization operators related to α-windowed Fourier transform”**, which was published in UPB Scientific Bulletin, Series A, Applied Mathematics and Physics, vol. 86, iss. 4, 2024, to which the author of the thesis was coauthor with Viorel Catană and Mihaela Grațiela Scumpu.

In the second section of Chapter 3, considering β a fast decreasing function in the Schwartz space $\mathcal{S}(\mathbb{R})$, we introduce in Definition 3. 4 a new class of localization operators $\beta L_{\sigma,\phi,\psi}^\alpha \overline{\beta} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ associated to the α-ferractal Fourier transform. We also study the properties of boundedness, compactness and belonging to Schatten-von Neumann classes for this class of localization operators.

Definition 3.1 *Let $\phi, \psi \in L^2(\mathbb{R}) \setminus \{0\}$ be two window functions and let α be a fractional parameter such that $\alpha \neq n\pi, n \in \mathbb{Z}$. Then, the operator $L_{\sigma,\phi,\psi}^\alpha : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ defined in weak sense by*

$$\begin{aligned}\left(L_{\sigma,\phi,\psi}^\alpha f, g\right)_{L^2(\mathbb{R})} &= \int_{\mathbb{R}}\int_{\mathbb{R}} \sigma(\omega,u) G_\phi^\alpha f(\omega,u)\overline{G_\psi^\alpha g(\omega,u)}d\omega du \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \sigma(\omega,u)\left(f, \phi_{\omega,u}^\alpha\right)_{L^2(\mathbb{R})}\left(\psi_{\omega,u}^\alpha, g\right)_{L^2(\mathbb{R})} d\omega du,\end{aligned}$$

for any $f, g \in L^2(\mathbb{R})$ or hard defined by

$$L^{\alpha}_{\sigma,\phi,\psi} f = \int_{\mathbb{R}} \int_{\mathbb{R}} \sigma(\omega, u) \left(f, \phi^{\alpha}_{\omega,u}\right)_{L^2(\mathbb{R})} \psi^{\alpha}_{\omega,u} d\omega du,$$

for any $f \in L^2(\mathbb{R})$, is called the localization operator associated to the α-windowed Fourier transform with respect to the symbol $\sigma \in L^1(\mathbb{R}^2) \cup L^{\infty}(\mathbb{R}^2)$.

Remark 3.1 Using the inversion formula from Theorem 2.6, the localization operator $L^{\alpha}_{\sigma,\phi,\psi} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ can be defined as

$$\left(L^{\alpha}_{\sigma,\phi,\psi} f, g\right)_{L^2(\mathbb{R})} = \frac{1}{2\pi \left(\psi, \phi\right)_{L^2(\mathbb{R})}} \int_{\mathbb{R}} \int_{\mathbb{R}} \sigma(\omega, u) G^{\alpha}_{\phi} f(\omega, u) \overline{G^{\alpha}_{\psi} g(\omega, u)} d\omega du,$$

for any $f, g \in L^2(\mathbb{R})$ and $\phi, \psi \in L^2(\mathbb{R}) \backslash \{0\}$ two window functions such that $(\psi, \phi)_{L^2(\mathbb{R})} \neq 0$.

We observe that if $\sigma(\omega, u) = 1$ for any $(\omega, u) \in \mathbb{R}^2$, then the orthogonality relation in Theorem 2.5 implies that the corresponding linear operator coincides with the identity operator on $L^2(\mathbb{R})$. So the function $\sigma : \mathbb{R}^2 \to \mathbb{C}$ which is called the symbol of the localization operator $L^{\alpha}_{\sigma,\phi,\psi} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is introduced to localize on $\mathbb{R}^2$, so that we obtain a non trivial bounded linear operator on $L^2(\mathbb{R})$. Hence the terminology frequently used by location operator.

It follows that the operators considered in Definition 3.1 and those in Remark 3.1 differs by a multiplicative constant that depends on the two admissible windows $\phi, \psi \in L^2(\mathbb{R}) \setminus \{0\}$.

In the following, we recall some results concerning the boundedness in $L^2(\mathbb{R})$ of the localization operator $L^{\alpha}_{\sigma,\phi,\psi} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$. Propositions 3.1 and 3.2 include the study of the boundedness property of the localization operator $L^{\alpha}_{\sigma,\phi,\psi}$ when the symbol σ belongs to the spaces $L^1(\mathbb{R}^2)$, respectively $L^{\infty}(\mathbb{R}^2)$.

Proposition 3.1 *Let $\sigma \in L^1(\mathbb{R}^2)$ be a symbol, let $\phi, \psi \in L^2(\mathbb{R}) \setminus \{0\}$ be two window functions and let α be a fractional parameter such that $\alpha \neq n\pi, n \in \mathbb{Z}$. Then, the localization operator $L^{\alpha}_{\sigma,\phi,\psi} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a well-defined bounded linear operator*

$$\left\| L^{\alpha}_{\sigma,\phi,\psi} \right\|_{\mathcal{B}(L^2(\mathbb{R}))} \leq \|\phi\|_{L^2(\mathbb{R})} \|\psi\|_{L^2(\mathbb{R})} \|\sigma\|_{L^1(\mathbb{R}^2)} .$$

Proposition 3.2 *Let $\sigma \in L^{\infty}(\mathbb{R}^2)$ be a symbol, let $\phi, \psi \in L^2(\mathbb{R}) \setminus \{0\}$ be two window functions and let α be a fractional parameter such that $\alpha \neq n\pi, n \in \mathbb{Z}$. Then, the localization operator $L^{\alpha}_{\sigma,\phi,\psi} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a well-defined bounded linear operator*

$$\left\| L^{\alpha}_{\sigma,\phi,\psi} \right\|_{\mathcal{B}(L^2(\mathbb{R}))} \leq 2\pi \|\phi\|_{L^2(\mathbb{R})} \|\psi\|_{L^2(\mathbb{R})} \|\sigma\|_{L^{\infty}(\mathbb{R}^2)} .$$

In the next theorem we study the boundedness in $L^2(\mathbb{R})$ of the localization operator $L^{\alpha}_{\sigma,\phi,\psi} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ when $\sigma \in L^p(\mathbb{R}^2), 1 \leq p \leq \infty$.

Theorem 3.1 *Let $\sigma \in L^p(\mathbb{R}^2), 1 \leq p \leq \infty$ be a symbol, let $\phi, \psi \in L^2(\mathbb{R}) \setminus \{0\}$ be two window functions and let α be a fractional parameter such that $\alpha \neq n\pi, n \in \mathbb{Z}$. Then, there exists a unique bounded linear operator $L^{\alpha}_{\sigma,\sigma,\phi,\psi} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ such that*

$$\left\|L^{\alpha}_{\sigma,\phi,\psi}\right\|_{\mathcal{B}(L^2(\mathbb{R}))} \leq (2\pi)^{\frac{1}{p'}} \|\phi\|_{L^2(\mathbb{R})} \|\psi\|_{L^2(\mathbb{R})} \|\sigma\|_{L^p(\mathbb{R}^2)},$$

where p' is the conjugate index of p $\left(i.e.\ \frac{1}{p} + \frac{1}{p'} = 1\right)$ and $L^{\alpha}_{\sigma,\phi,\psi}$ is defined for any $f, g \in L^2(\mathbb{R})$ and all simple functions $\sigma \in \mathbb{R}^2$ for which $\mu\{(\omega, u) \in \mathbb{R}^2 : \sigma(\sigma(\omega, u) \neq 0\} < \infty$.

In the following statements we present some results concerning the Schatten-von Neumann properties of the localization operator $L^{\alpha}_{\sigma,\phi,\psi} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, when its symbol σ is in $L^p(\mathbb{R}^2), 1 \leq p \leq \infty$.

Proposition 3.3 *Let $\sigma \in L^1(\mathbb{R}^2)$ be a symbol, let $\phi, \psi \in L^2(\mathbb{R}) \setminus \{0\}$ be two window functions and let α be a fractional parameter such that $\alpha \neq n\pi, n \in \mathbb{Z}$. Then, the localization operator $L^{\alpha}_{\sigma,\phi,\psi} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is in the Hilbert-Schmidt class S_2 and*

$$\left\|L^{\alpha}_{\sigma,\phi,\psi}\right\|^2_{S_2} = \sum_{n=1}^{\infty} \left\|L^{\alpha}_{\sigma,\phi,\psi}\xi_n\right\|^2_{L^2(\mathbb{R})},$$

where $\{\xi_n\}_{n\geq 1}$ is an orthonormal basis of $L^2(\mathbb{R})$.

Proposition 3.4 *Let $\sigma \in L^1(\mathbb{R}^2)$ be a symbol, let $\phi, \psi \in L^2(\mathbb{R}) \setminus \{0\}$ be two window functions and let α be a fractional parameter such that $\alpha \neq n\pi, n \in \mathbb{Z}$. Then, the localization operator $L^{\alpha}_{\sigma,\phi,\psi} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is in the trace class S_1 and*

$$\left\|L^{\alpha}_{\sigma,\phi,\psi}\right\|_{S_1} \leq \|\phi\|_{L^2(\mathbb{R})} \|\psi\|_{L^2(\mathbb{R})} \|\sigma\|_{L^1(\mathbb{R}^2)}.$$

In the next proposition we state and prove a compactness result about the localization operator $L^{\alpha}_{\sigma,\phi,\psi} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, under the hypothesis that its symbol $\sigma \in L^p(\mathbb{R}^2), 1 \leq p < \infty$.

Proposition 3.5 *Let $\sigma \in L^p(\mathbb{R}^2), 1 \leq p < \infty$ be a symbol, let $\phi, \psi \in L^2(\mathbb{R}) \setminus \{0\}$ be two window functions and let α be a fractional parameter such that $\alpha \neq n\pi, n \in \mathbb{Z}$. Then, the localization operator $L^{\alpha}_{\sigma,\phi,\psi} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is compact.*

Theorem 3.4 *Let $\sigma \in L^p(\mathbb{R}^2), 1 \leq p \leq \infty$ be a symbol, let $\phi, \psi \in L^2(\mathbb{R}) \setminus \{0\}$ be two window functions and let α be a fractional parameter such that $\alpha \neq n\pi, n \in \mathbb{Z}$. Then, the localization operator $L^{\alpha}_{\sigma,\phi,\psi} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is in the Schatten-von Neumann class S_p and*

$$\left\|L^{\alpha}_{\sigma,\phi,\psi}\right\|_{S_p} \leq (2\pi)^{\frac{1}{p'}} \|\phi\|_{L^2(\mathbb{R})} \|\psi\|_{L^2(\mathbb{R})} \|\sigma\|_{L^p(\mathbb{R}^2)},$$

where p' is the conjugate index of p.

In the next result we present a bilateral estimate of the norm in the trace class of the localization operator $L^{\alpha}_{\sigma,\phi,\psi} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, considering the symbol $\sigma \in L^1(\mathbb{R}^2)$.

Theorem 3.5 *Let $\sigma \in L^1(\mathbb{R}^2)$ be a symbol, let $\phi, \psi \in L^2(\mathbb{R}) \setminus \{0\}$ be two window functions and let α be a fractional parameter such that $\alpha \neq n\pi,\ n \in \mathbb{Z}$. Then, the* *following estimation holds*

$$\frac{1}{\pi\left(\|\phi\|^2_{L^2(\mathbb{R})}+\|\psi\|^2_{L^2(\mathbb{R})}\right)}\|\sigma_{\phi,\psi}\|_{L^1(\mathbb{R}^2)} \le \left\|L^\alpha_{\sigma,\phi,\psi}\right\|_{S_1} \le \|\phi\|_{L^2(\mathbb{R})}\|\psi\|_{L^2(\mathbb{R})}\|\sigma\|_{L^1(\mathbb{R}^2)}, \tag{15}$$

where $\sigma_{\phi,\psi}:\mathbb{R}^2\to\mathbb{C}$ *is defined by*

$$\sigma_{\phi,\psi}(\omega,u)=\left(L^\alpha_{\sigma,\phi,\psi}\phi^\alpha_{\omega,u},\psi^\alpha_{\omega,u}\right)_{L^2(\mathbb{R})}.$$

In the next proposition we establish a formula for the trace $tr\left(L^\alpha_{\sigma,\phi,\psi}\right)$ of the localization operator $L^\alpha_{\sigma,\phi,\psi}:L^2(\mathbb{R})\to L^2(\mathbb{R})$ under the assumption that $\sigma\in L^1(\mathbb{R}^2)$.

Proposition 3.6 *Under the hypotheses of Theorem 3.4 it follows that the trace* $tr\left(L^\alpha_{\sigma,\phi,\psi}\right)$ *of the localization operator* $L^\alpha_{\sigma,\phi,\psi}$ *is given by the formula*

$$tr\left(L^\alpha_{\sigma,\phi,\psi}\right)=\int_{\mathbb{R}}\int_{\mathbb{R}}\sigma(\omega,u)\left(\psi^\alpha_{\omega,u},\phi^\alpha_{\omega,u}\right)_{L^2(\mathbb{R})}d\omega du.$$

Remark 3.2 If σ is a positive real function in $L^1(\mathbb{R}^2)$ and $\phi=\psi$, then the estimates in relation (15) are sharp.

In the following we state and prove a result regarding the norm in the trace class S_1 of the power n for the product of two localization operators.

Proposition 3.7 *Let* σ_1,σ_2 *be two positive real functions in* $L^1(\mathbb{R}^2)$ *and let* ϕ *be a window function in* $L^2(\mathbb{R})\setminus\{0\}$. *Suppose that the operators* $L^\alpha_{\sigma_1,\phi,\phi}:L^2(\mathbb{R})\to L^2(\mathbb{R})$ *and* $L^\alpha_{\sigma_2,\phi,\phi}:L^2(\mathbb{R})\to L^2(\mathbb{R})$ *commute with each other, and the operator* $L^\alpha_{\sigma_1,\phi,\phi}L^\alpha_{\sigma_2,\phi,\phi}:L^2(\mathbb{R})\to L^2(\mathbb{R})$ *is positive. Then, the linear operators* $L^\alpha_{\sigma_1,\phi,\phi}$, $L^\alpha_{\sigma_2,\phi,\phi}$ *and* $L^\alpha_{\sigma_1,\phi,\phi}L^\alpha_{\sigma_2,\phi,\phi}$ *are positive and belong to the trace class* S_1. *Moreover,*

$$\left\|\left(L^\alpha_{\sigma_1,\phi,\phi}L^\alpha_{\sigma_2,\phi,\phi}\right)^n\right\|_{S_1}\le\left\|L^\alpha_{\sigma_1,\phi,\phi}\right\|^n_{S_1}\left\|L^\alpha_{\sigma_2,\phi,\phi}\right\|^n_{S_1},$$

for all $n\in\mathbb{N}$.

In subchapter 3.2 we focus our attention on a new class of localization operators of the form $\beta L^\alpha_{\sigma,\phi,\psi}\overline{\beta}:L^2(\mathbb{R})\to L^2(\mathbb{R})$ associated to the α-windowed Fourier transform, considering β a fast decreasing function in the Schwartz space $\mathcal{S}(\mathbb{R})$.

In the next result we define a new class of localization operators of the form $\beta L^\alpha_{\sigma,\phi,\psi}\overline{\beta}:L^2(\mathbb{R})\to L^2(\mathbb{R})$ associated to the α-windowed Fourier transform.

Definition 3.9 *Let* $\sigma\in L^1(\mathbb{R}^2)+L^\infty(\mathbb{R}^2)$ *be a symbol, let* $\phi,\psi\in L^2(\mathbb{R})\setminus\{0\}$ *be two admissible window functions (wavelets), let* α *be a fractional parameter such that* $\alpha\neq n\pi, n\in\mathbb{Z}$ *and let* $\beta\in\mathcal{S}(\mathbb{R})$, *where* $\mathcal{S}(\mathbb{R})$ *is the Schwartz space of rapidly decreasing functions on* $\mathbb{R}$. *Then, the localization operator associated to* α*-windowed*

Fourier transform $\beta L^{\alpha}_{\sigma,\phi,\psi}\overline{\beta} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ *is defined by*

$$\begin{aligned}\left(\beta L^{\alpha}_{\sigma,\phi,\psi}\overline{\beta}f, g\right)_{L^2(\mathbb{R})} &= \left(L^{\alpha}_{\sigma,\phi,\psi}\overline{\beta}f, \overline{\beta}g\right)_{L^2(\mathbb{R})} \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \sigma(\omega,u)\left(\overline{\beta}f, \phi^{\alpha}_{\omega,u}\right)_{L^2(\mathbb{R})}\left(\psi^{\alpha}_{\omega,u}, \overline{\beta}g\right)_{L^2(\mathbb{R})} d\omega du \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \sigma(\omega,u)\left(f, \beta\phi^{\alpha}_{\omega,u}\right)_{L^2(\mathbb{R})}\left(\beta\psi^{\alpha}_{\omega,u}, g\right)_{L^2(\mathbb{R})} d\omega du,\end{aligned} \tag{16}$$

for any $f, g \in L^2(\mathbb{R}^2)$.

These non-adjoint operators are reminiscent of the Landau-Pollak-Slepian operators encountered in papers [22], [27], [30], [31] and [32].

For the operators introduced in the relation (16) we are interested to study the Schatten-von Neumann boundary and properties. A first result in this sense is related to the L^2-boundedness of the localization operator $\beta L^{\alpha}_{\sigma,\phi,\psi}\overline{\beta} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ when the symbol $\sigma \in L^1(\mathbb{R}^2)$.

Proposition 3.8 *Let* $\sigma \in L^1(\mathbb{R}^2)$ *be a symbol, let* $\phi, \psi \in L^2(\mathbb{R}) \setminus \{0\}$ *be two admissible window functions and let* α *be a fractional parameter such that* $\alpha \neq n\pi, n \in \mathbb{Z}$. *Then, for any function* β *in the Schwartz space* $\mathcal{S}(\mathbb{R})$, *the localization operator* $\beta L^{\alpha}_{\sigma,\phi,\psi}\overline{\beta} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ *is bounded and*

$$\left\|\beta L^{\alpha}_{\sigma,\phi,\psi}\overline{\beta}\right\|_{\mathcal{B}(L^2(\mathbb{R}))} \leq \|\beta\|^2_{L^{\infty}(\mathbb{R})} \|\phi\|_{L^2(\mathbb{R})} \|\psi\|_{L^2(\mathbb{R})} \|\sigma\|_{L^1(\mathbb{R}^2)}.$$

A result concerning the L^2-boundedness of the localization operator $\beta L^{\alpha}_{\sigma,\phi,\psi}\overline{\beta} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, when the symbol $\sigma \in L^{\infty}(\mathbb{R}^2)$, is presented in the following proposition.

Proposition 3.9 *Let* $\sigma \in L^{\infty}(\mathbb{R}^2)$ *be a symbol, let* $\phi, \psi \in L^2(\mathbb{R}) \setminus \{0\}$ *be two admissible window functions and let* α *be a fractional parameter such that* $\alpha \neq n\pi, n \in \mathbb{Z}$. *Then, for any function* β *in the Schwartz space* $\mathcal{S}(\mathbb{R})$, *the localization operator* $\beta L^{\alpha}_{\sigma,\phi,\psi}\overline{\beta} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ *is bounded and*

$$\left\|\beta L^{\alpha}_{\sigma,\phi,\psi}\overline{\beta}\right\|_{\mathcal{B}(L^2(\mathbb{R}))} \leq 2\pi \|\beta\|^2_{L^{\infty}(\mathbb{R})} \|\phi\|_{L^2(\mathbb{R})} \|\psi\|_{L^2(\mathbb{R})} \|\sigma\|_{L^{\infty}(\mathbb{R}^2)}.$$

In the next theorem we present a boundedness result in $L^2(\mathbb{R})$ for the localization operator $\beta L^{\alpha}_{\sigma,\phi,\psi}\overline{\beta} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ when the symbol $\sigma \in L^p(\mathbb{R}^2)$, $1 \leq p \leq \infty$.

Theorem 3.6 *Let* $\sigma \in L^p(\mathbb{R}^2)$, $1 \leq p \leq \infty$ *be a symbol, let* $\phi, \psi \in L^2(\mathbb{R}) \setminus \{0\}$ *be two admissible window functions and let* α *be a fractional parameter such that* $\alpha \neq n\pi, n \in \mathbb{Z}$. *Then, for any fixed function* β *in the Schwartz space* $\mathcal{S}(\mathbb{R})$, *there exists a unique bounded linear localization operator* $\beta L^{\alpha}_{\sigma,\phi,\psi}\overline{\beta} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ *such that*

$$\left\|\beta L^{\alpha}_{\sigma,\phi,\psi}\overline{\beta}\right\|_{\mathcal{B}(L^2(\mathbb{R}))} \leq (2\pi)^{\frac{1}{p'}} \|\beta\|^2_{L^{\infty}(\mathbb{R})} \|\phi\|_{L^2(\mathbb{R})} \|\psi\|_{L^2(\mathbb{R})} \|\sigma\|_{L^p(\mathbb{R}^2)},$$

where p' *is the conjugate index of* p $\left(\textit{i.e. } \frac{1}{p} + \frac{1}{p'} = 1\right)$.

We can also state and prove a result regarding the belonging of the localization operator $\beta L^{\alpha}_{\sigma,\phi,\psi}\overline{\beta} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ to the trace class S_1.

Theorem 3.7 *Let $\sigma \in L^1(\mathbb{R}^2)$ be a symbol, let $\phi, \psi \in L^2(\mathbb{R}) \setminus \{0\}$ be two admissible window functions and let α be a fractional parameter such that $\alpha \neq n\pi, n \in \mathbb{Z}$. Then, for any function β in the Schwartz space $\mathcal{S}(\mathbb{R})$, the localization operator $\beta L^{\alpha}_{\sigma,\phi,\psi}\overline{\beta} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is in the trace class S_1 and*

$$\left\|\beta L^{\alpha}_{\sigma,\phi,\psi}\overline{\beta}\right\|_{\mathcal{S}_1} \leq \|\beta\|^4_{L^\infty(\mathbb{R})} \|\phi\|_{L^2(\mathbb{R})} \|\psi\|_{L^2(\mathbb{R})} \|\sigma\|_{L^1(\mathbb{R}^2)}.$$

In the following result we present the formula for the trace $\operatorname{tr}\left(\beta L^{\alpha}_{\sigma,\phi,\psi}\overline{\beta}\right)$ of the localization operator $\beta L^{\alpha}_{\sigma,\phi,\psi}\overline{\beta} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$.

Proposition 3.10 *The trace $tr\left(\beta L^{\alpha}_{\sigma,\phi,\psi}\overline{\beta}\right)$ of the localization operator $\beta L^{\alpha}_{\sigma,\phi,\psi}\overline{\beta} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is given by*

$$tr\left(\beta L^{\alpha}_{\sigma,\phi,\psi}\overline{\beta}\right) = \int_{\mathbb{R}} \int_{\mathbb{R}} \sigma(\omega, u) \left(\beta \psi^{\alpha}_{\omega,u}, \beta \phi^{\alpha}_{\omega,u}\right)_{L^2(\mathbb{R})} d\omega du.$$

A result concerning the compactness of the localization operator $\beta L^{\alpha}_{\sigma,\phi,\psi}\overline{\beta} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is presented in the following.

Proposition 3.11 *Let $\sigma \in L^p(\mathbb{R}^2)$, $1 \leq p \leq \infty$ be a symbol, let $\phi, \psi \in L^2(\mathbb{R}) \setminus \{0\}$ be two admissible window functions and let α be a fractional parameter such that $\alpha \neq n\pi, n \in \mathbb{Z}$. Then, for any function β in the Schwartz space $\mathcal{S}(\mathbb{R})$, the localization operator $\beta L^{\alpha}_{\sigma,\phi,\psi}\overline{\beta} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is compact.*

Considering the symbol $\sigma \in L^p(\mathbb{R}^2)$, $1 \leq p \leq \infty$ and using the Riez-Thorin interpolation theorem for Lebesgue spaces and Schatten-von Neumann classes it can be proven that the localization operator $\beta L^{\alpha}_{\sigma,\phi,\psi}\overline{\beta} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is in the Schatten-von Neumann class S_p.

Proposition 3.12 *Let $\sigma \in L^p(\mathbb{R}^2)$, $1 \leq p \leq \infty$ be a symbol, let $\phi, \psi \in L^2(\mathbb{R}) \setminus \{0\}$ be two admissible window functions and let α be a fractional parameter such that $\alpha \neq n\pi, n \in \mathbb{Z}$. Then, for any function β in the Schwartz space $\mathcal{S}(\mathbb{R})$, the localization operator $\beta L^{\alpha}_{\sigma,\phi,\psi}\overline{\beta} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is in the Schatten-von Neumann class S_p and*

$$\left\|\beta L^{\alpha}_{\sigma,\phi,\psi}\overline{\beta}\right\|_{S_p} \leq (2\pi)^{\frac{1}{p'}} \|\beta\|^2_{L^\infty(\mathbb{R})} \|\phi\|_{L^2(\mathbb{R})} \|\psi\|_{L^2(\mathbb{R})} \|\sigma\|_{L^p(\mathbb{R}^2)},$$

where p' is the conjugate index of p.

In the next result we present a bilateral estimate of the norm in the trace class of the localization operator $\beta L^{\alpha}_{\sigma,\phi,\psi}\overline{\beta} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, considering the symbol $\sigma \in L^1(\mathbb{R}^2)$.

Theorem 3.8 *Let $\sigma \in L^1(\mathbb{R}^2)$ be a symbol, let $\phi, \psi \in L^2(\mathbb{R}) \setminus \{0\}$ be two admissible window functions and let α be a fractional parameter such that $\alpha \neq n\pi, n \in \mathbb{Z}$. Then,* *the following estimate holds*

$$\frac{1}{\pi\left(\|\phi\|_{L^2(\mathbb{R})}^2+\|\psi\|_{L^2(\mathbb{R})}^2\right)}\|\sigma_{\phi,\psi}\|_{L^1(\mathbb{R}^2)} \le \left\|\beta L^{\alpha}_{\sigma,\phi,\psi}\overline{\beta}\right\|_{\mathcal{S}_1}$$
$$\le \|\beta\|_{L^\infty(\mathbb{R})}^2 \|\phi\|_{L^2(\mathbb{R})} \|\psi\|_{L^2(\mathbb{R})} \|\sigma\|_{L^1(\mathbb{R}^2)},$$

unde $\sigma_{\phi,\psi}:\mathbb{R}^2\to\mathbb{C}$ *is given by*

$$\sigma_{\phi,\psi}(\omega,u)=\left(\beta L^{\alpha}_{\sigma,\phi,\psi}\overline{\beta}\phi^{\alpha}_{\omega,u},\psi^{\alpha}_{\omega,u}\right)_{L^2(\mathbb{R})}.$$

Selected bibliography

[1] L.B. Almeida, *The fractional Fourier transform and time-frequency representations*, IEEE Trans. Signal Process., vol. 42, 3084–3091, 1994.

[2] M. Bahri, F.A. Shah, A.Y. Tantary, *Uncertainty principles for the continuous shearlet transforms in arbitrary space dimensions*, Integ. Transf. Special Funct, 2020, https://doi.org/10.1080/10652469.2019.1707816.

[3] M. Bravo, H. Prado, E.G. Reyes, *Nonlinear pseudo-differential equations defined by elliptic symbols on $L^p(\mathbb{R}^n)$ and the fractional Laplacian*, arXiv:1611.09277v2[math.AP], 2018.

[4] M. Bravo-Vera, *Nonlinear equations of infinite order defined by an elliptic symbol*, Int. J. Math. Math. Sci., vol. 2014, Article ID656959, 7 pages.

[5] V. Catană, *Products of Two-Wavelet Multipliers and Their Traces*, Pseudo-Differential Operators: Complex Analysis and Partial Differential Equations, Operator Theory: Advances and Applications, vol. 205, 195–211, Birkhäuser, Verlag, Basel, 2010.

[6] V. Catană, *Schatten–von Neumann norm inequalities for two-wavelet localization operators*, Pseudo-Differential Operators: Partial Differential Equations and Time-Frequency Analysis, Fields Institute Communications, vol. 52, 265–277, American Mathematical Society, 2007.

[7] V. Catană, *S-operators related to a finite measure space*, Appl. Anal., vol. 97, 326-339, 2018.

[8] V. Catană, *Z-operators related to a finite measure space*, J. Pseudo-Differ. Oper. Appl., vol. 9, 173-188, 2018.

[9] V. Catană, **I.M. Flondor**, H.G. Georgescu, *Generalized Fourier multipliers*, Ann. Funct. Anal., vol. 14, 34, 2023.

[10] V. Catană, **I.M. Flondor**, M.G. Scumpu, *α-Windowed Fourier transform (α-WFT)*, J. Pseudo-Differ. Oper. Appl., vol.15, 75, 2024.

[11] V. Catană, **I.M. Flondor**, M.G. Scumpu, *Localization operators related to α-windowed Fourier transform*, U.P.B. Sci. Bull., Series A, vol. 86, iss. 4, 2024.

[12] V. Catană, H.G. Georgescu, **I.M. Flondor**, *On a generalized class of nonlinear equations defined by elliptic symbols*, Bull. Malays. Math. Sci. Soc., vol. 47, article number 109, 2024.

[13] I. Daubechies, *Ten Lectures on Wavelets*. SIAM, Philadelphia, 1992.

[14] J. Du, M.W. Wong, *Traces of wavelet multipliers*, C. R. Math. Rep. Acad. Sci. Canada, vol. 23, 148-152, 2001.

[15] D. Gabor, *Theory of communications*, J. Inst. Electr. Eng., vol. 93, 429–457, 1946.

[16] P. Górka, H. Prado, E.G. Reyes, *On a general class of nonlocal equations*, Ann. Henri Poincaré, vol. 14, 947-966, 2013.

[17] K. Gröchenig, *Foundation of Time-Frequency Analysis*, Birkhäuser, Boston, 2001.

[18] X. Guanlei, W. Xiaotong, X. Xiaogang, *New inequalities and uncertainty relations on linear canonical transform revisit*, EURASIP J. Adv. Signal Process., Article ID: 563265, 7 pp, 2009.

[19] X. Guanlei, W. Xiaotong, X. Xiaogang, *Uncertainty inequalities for linear canonical transform*, IET Signal Process., vol. 3(5), 392–402, 2009.

[20] Z. He, M.W. Wong, *Wavelet multipliers and signals*, J. Austral. Math. Ser. B, vol. 40, 437-446, 1999.

[21] K.I. Kou, R.H. Xu, Y.H. Zhang, *Paley–Wiener theorems and uncertainty principles for the windowed linear canonical transform*, Math. Methods Appl. Sci., vol. 35, 2122–2132, 2012.

[22] H.J. Landau, H. O. Pollak, *Prolate spheroidal wave functions, Fourier analysis and uncertainty*, II, Bell Syst. Tech. J., vol. 40, 65-84, 1961.

[23] S. Molahajloo, *Pseudo-differential operators on* $\mathbb{Z}$, B.-W. Schultze, M.W. Wong (eds.) Pseudo-Differential Operators: Complex Analysis and Partial Differential Equations, Birkhäuser, Basel, 213–221, 2010.

[24] S. Molahajloo, M.W. Wong, *Discrete Analogs of Wigner Transforms and Weyl Transforms*, Published in: Analysis of Pseudo-Differential Operators, Editors: S. Molahajloo, M.W. Wong, 1-20, Birkhäuser, 2019.

[25] S. Molahajloo, M.W. Wong, *Pseudo-differential operators on* S^1, L. Rodino, M.W. Wong (eds.) New Developments in Pseudo-Differential Operators, Birkhäuser, Basel, 297–306, 2008.

[26] S. Molahajloo, M.W. Wong, *Ellipticity, Fredholmness and spectral invariance of pseudo-differential operators on* S^1, J. Pseudo-Differ. Oper. Appl., vol. 1, 183–205, 2010.

[27] H.O. Pollak, *Prolate spheroidal wave functions, Fourier analysis and uncertainty*, III, Bell Syst. Tech. J., vol. 41, 1295-1336, 1962.

[28] B.P. Rynne, M.A. Youngson, *Linear Functional Analysis*, Second Edition, Springer-Verlag, 2008.

[29] M.G. Scumpu, PhD thesis: *Transformări integrale timp-frecvență și operatori de localizare asociați acestor transformări (Time-frequency integral transforms and associated localization operators)*, Bucharest, 2024.

[30] D. Slepian, *On bandwidth*, Proc. IEEE, vol. 64, 292-300, 1976.

[31] D. Slepian, *Some comments on Fourier analysis, uncertainty and modeling*, SIAM Rev., vol. 25, 379-393, 1983.

[32] D. Slepian, H. O. Pollak, *Prolate spheroidal wave functions, Fourier analysis and uncertainty*, I, Bell Syst. Tech. J., vol. 40, 43-64, 1961.

[33] H.M. Srivastava, A.F. Shah, A.Y. Tantary, *A family of convolution-based generalized Stockwell transforms*, J. Pseudo-Differ. Oper., vol. 11, 1505–1536, 2020.

[34] R. Tao, Y.L. Li, Y. Wang, *Uncertainty principles for linear canonical transforms*, IEEE Trans. Signal Process., vol. 57(7), 2856–2858, 2009.

[35] M.W. Wong, *Discrete Fourier Analysis*, Birkhäuser, 2011.

[36] M.W. Wong, *Wavelet Transforms and Localization Operators*, Birkhäuser, Boston, 2002, (Operator theory; Advances and Applications; vol. 136).

[37] M.W. Wong, *Weyl Transforms*, Springer-Verlag, New-York, Berlin-Heidelberg, 1998.

[38] M.W. Wong, Z. Zhang, *Trace class norm inequalities for wavelet multipliers*, Bulletin of the London Mathematical Society, vol. 34(6), 739-744, 2002.

[39] Y. Yang, K.I. Kou, *On uncertainty principles for hypercomplex signals in the linear canonical transform domains*, Signal Process., vol. 95, 67–75, 2014.

[40] K. Zhu, *Operator Theory in Function Spaces*, Marcel Dekker Inc., Pure and Applied Mathematics a Series of Monographs and Textbooks, vol. 139, New York, USA, 1990.